Entanglement reciprocation between qubits and continuous variables

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We investigate how entanglement can be transferred between qubits and continuous variable (CV) systems. We find that one ebit borne in maximally entangled qubits can be fully transferred to two CV systems which are initially prepared in pure separable Gaussian field with high excitation. We show that it is possible, though not straightforward, to retrieve the entanglement back to qubits from the entangled CV systems. The possibility of deposition of multiple ebits from qubits to the initially unentangled CV systems is also pointed out.

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Quantum information processing (QIP) has been extensively studied for a qubit system which is a quantum extension of a bit, spanning two-dimensional Hilbert space. A qubit is realized by a spin, a two-level atom, the polarization of a photon and a superconductor among others. A two-dimensional system is mathematically handy and logically easy to treat. On the other hand, many continuous-variable (CV) physical systems such as a harmonic oscillator and a light field, which are defined in infinite-dimensional Hilbert space, have also attracted considerable attention for other practical reasons. While qubit and CV systems are nearly always treated separately, there is a good reason to believe that a study of their interface may result in synergy for the implementation of the QIP. There have been some pilot works on how to entangle two separate qubits by an entangled Gaussian field [1, 2, 3]. In this paper, we ask the interesting questions of how easy it is to: a) deposit the entanglement of two qubits to a pool of coherent states and b) retrieve quantum entanglement back to qubits from the pool.

When two maximally entangled two-level atoms are sent to two respective cavities, initially prepared in vacuum, after the Rabi time the maximal entanglement is fully transferred to the cavity fields [4, 5]. Here the interaction is assumed resonant and the cavities are lossless. Essentially, in the above transfer the cavity does not behave as a true CV system, as only the $|0\rangle$ and $|1\rangle$ states of cavity take part. The cavity initially in the vacuum is like a pool without a drop of water so that dropping a tiny bit of water will be noticeable. However, if the pools are full of water, an additional drop of water will not make a difference. This means that there is a chance that when the cavities are prepared with coherent fields of large amplitudes, atoms' depositing extra excitation will probably not make a big difference and the cavity fields will not be entangled much (or even at all). How about a possibility to retrieve the quantum entanglement by the second set of atoms which interact with the cavity fields left by the first set of atoms? Will the atoms be able to recover the entanglement deposited by the first set of atoms? In this paper, we find an answer to these questions.

Model.- Let us consider two atoms in the triplet state

$$|\psi(0)\rangle_a = \frac{1}{\sqrt{2}}\left(|e\rangle_1|g\rangle_2 + |g\rangle_1|e\rangle_2\right), \tag{1}$$

where $|e\rangle$ and $|g\rangle$ stand for the excited and ground states of the atom. This state is maximally entangled and is said to carry one *ebit* of entanglement. The two atoms enter their respective cavities which are initially prepared with coherent states. For convenience, we assume that the amplitudes of the coherent states are same to α. The initial state for the atoms and fields is $|\Psi(0)\rangle_{af} = |\psi(0)\rangle_a|\alpha\rangle_1|\alpha\rangle_2$.

We consider how much the atomic entanglement, in units of ebit, is transferred to the infinite-dimensional cavity fields by their resonant interaction. Under the rotating wave approximation, the interaction Hamiltonian $\hat{H} = \hbar\lambda(\hat{a}^\dagger|g\rangle\langle e| + \hat{a}|e\rangle\langle g|)$. The bosonic creation and annihilation operators are denoted by $\hat{a}^\dagger$ and $\hat{a}$, respectively, the coupling between the field and the atom by λ and t is the duration of interaction. In this case, the evolution of the atom and field state is determined by the following propagation operator: $\hat{U} = \hat{U}_1 \otimes \hat{U}_2$ where, in atomic atomic bases $\langle e| = (1,0)$ and $\langle g| = (0,1)$,

$$\hat{U}_i = \begin{pmatrix} \hat{U}_{11}^{(i)} & \hat{U}_{12}^{(i)} \\ \hat{U}_{21}^{(i)} & \hat{U}_{22}^{(i)} \end{pmatrix} \tag{2}$$

with the operators [6]

$$\begin{aligned} \hat{U}_{11}^{(i)} &= \cos\lambda t\sqrt{\hat{a}_i\hat{a}_i^\dagger}, \qquad \hat{U}_{12}^{(i)} = -i\hat{a}_i\frac{\sin\lambda t\sqrt{\hat{a}_i^\dagger\hat{a}_i}}{\sqrt{\hat{a}_i^\dagger\hat{a}_i}}, \\ \hat{U}_{21}^{(i)} &= -i\hat{a}_i^\dagger\frac{\sin\lambda t\sqrt{\hat{a}_i\hat{a}_i^\dagger}}{\sqrt{\hat{a}_i\hat{a}_i^\dagger}}, \qquad \hat{U}_{22}^{(i)} = \cos\lambda t\sqrt{\hat{a}_i^\dagger\hat{a}_i} \end{aligned} \tag{3}$$

where the subscript of $\hat{a}$ and $\hat{a}^\dagger$ denotes the mode of the field.

After the interaction, the atom-field state evolves to $\hat{U}|\Psi(0)\rangle_{a-f}$. Here, we postselect the cavity field conditioned on two atoms leaving the cavities in their ground states. The main reason of the postselection is to bring the cavity field to a pure state, whose measure of entanglement is the von Neumann entropy of the reduced density operator. While in this paper we are interested in a possibility for qubits to deposit one complete ebit to a large CV system, there is no measure or criterion of entanglement for a general CV state. The field state after postselection is

$$|\psi(1)\rangle_f = \frac{\mathcal{N}}{\sqrt{2}}\left(\hat{U}^{(1)}_{21}\hat{U}^{(2)}_{22} + \hat{U}^{(1)}_{22}\hat{U}^{(2)}_{21}\right)|\alpha\rangle_1|\alpha\rangle_2 \tag{4}$$

The normalization constant is denoted by $\mathcal{N}$ and the coherent state is expanded [7] such as $|\alpha\rangle = \sum_m C_m|m\rangle$ where $C_m = \alpha^m e^{-\frac{1}{2}|\alpha|^2}/\sqrt{m!}$ gives a Poissonian weight with the average photon number $\bar{n} = |\alpha|^2$. Substituting these into Eq. (4), we find

$$|\psi(1)\rangle_f = \sum_{n,m=0}^{\infty} \mathcal{C}_{n,m}|n\rangle_1|m\rangle_2 \tag{5}$$

with $\mathcal{C}_{n,m} = \frac{-i\mathcal{N}e^{-|\alpha|^2}}{\alpha\sqrt{2}}[\frac{\alpha^{n+m}\sin(\lambda t\sqrt{n})\cos(\lambda t\sqrt{m})}{\sqrt{m!(n-1)!}} + n \leftrightarrow m]$.

Entanglement transfer from qubits to CV.- The atoms initially have one ebit as they are maximally entangled. We like to know how much ebit is transferred to the cavity fields by the resonant interaction. As the cavity fields are in a pure state $|\psi(1)\rangle_f$, the amount of ebit $\mathcal{E}$ is calculated by $\mathcal{E} = -\mathrm{Tr}\hat{\rho}_{f1}\log_2\hat{\rho}_{f1}$ where the reduced density operator for the cavity field 1 is

$$\hat{\rho}_{f1} = \mathrm{Tr}_{f2}\hat{\rho}_f = \sum_{m,n,n'} \mathcal{C}_{n,m}\mathcal{C}^*_{n',m}|n\rangle\langle n'|. \tag{6}$$

In Fig.1 **(a)** we plot $\mathcal{E}$ against α and the interaction time λt (unit of π). When $\alpha = 0$, we know $\mathcal{E} = 1$ for sure. Fig.1 **(b)** shows that the probability of the atoms leaving the cavities in the ground states. When $\alpha < 1$, an oscillating behavior is observed in the degree of entanglement as well as in the atomic population. On the other hand, it is interesting to note that when α is large the cavities are with complete ebit whenever the atoms leave the cavities in their ground states except the first moments of oscillations. We can analyze this by showing that $|\phi_1\rangle \equiv \hat{U}_{21}|\alpha\rangle$ is orthogonal to $|\phi_2\rangle \equiv \hat{U}_{22}|\alpha\rangle$ in Eq. (4). In other words

$$v_0 \equiv |\langle\phi_1|\phi_2\rangle| = \frac{e^{-|\alpha|^2}}{2}\sum_n \frac{\sqrt{n}}{\alpha}\frac{\alpha^{2n}}{n!}\sin 2\lambda t\sqrt{n} \tag{7}$$

has to be zero. If so, state (4) becomes a maximally entangled qubit state as it will be an equally weighted superposition of two orthogonal composite states. For simplicity, let us take α real. In the limit of $\alpha^2 \gg 1$, the Poissonian distribution is replaced by a Gaussian distribution over the variable n with mean value and variance equal to α^2 [7] so that

$$C_n^2 \equiv e^{-\alpha^2}\frac{\alpha^{2n}}{n!} \approx \frac{1}{\sqrt{2\pi\alpha^2}}e^{-\frac{(n-\alpha^2)^2}{2\alpha^2}}. \tag{8}$$

Taking into account the largely contributing terms of $\sqrt{n}$, *i.e.* those of n near the peak α^2, we have $\sqrt{n} = \sqrt{\alpha^2 + (n-\alpha^2)} \approx \alpha\left(1 + \frac{n-\alpha^2}{2\alpha^2}\right)$. Finally, the summation over n is replaced by an integration in terms of $x = (n-\alpha^2)/\alpha$ and the integration region is extended to $(-\infty, \infty)$. We then immediately recognize v_0 as a Fourier transformation of a Gaussian function:

$$v_0 \propto (\sin 2\alpha\lambda t + \frac{\lambda t}{2\alpha}\cos 2\lambda\alpha t)e^{-\frac{\lambda^2 t^2}{2}}, \tag{9}$$

which decreases exponentially to zero and the two states become orthogonal to each other exponentially with regard to the interaction time. This shows the transfer of a complete ebit from two qubits to a CV system of a large amplitude. It is straightforward to show that the probability of the postselection is 1/4 for the limit considered here.

Using the same analogy to prove their orthogonality, we can show that $\langle\phi_0|\phi_0\rangle = \langle\phi_1|\phi_1\rangle$ for $\alpha \gg 1$. Suppose the initial atomic state was prepared not in the perfect triplet state (1) but in a partially entangled mixed or pure state. If we again assume the case of postselecting atoms in their ground states, from the earlier analysis we know that the atom initially in $|e\rangle$ will take the initial coherent field to $|\phi_1\rangle$ and $|g\rangle$ to $|\phi_2\rangle$. As the two field state bases are orthogonal with the same weight, it is straightforward to show that the field state collapses to the state which bears the same amount of entanglement as in the initial atomic qubits. This shows the perfect transfer of initial entanglement to a CV system.

In order to see the transfer of the ebit, we took a limit to ignore the discrete nature of photons. However, it is interesting to note that we need to recover the

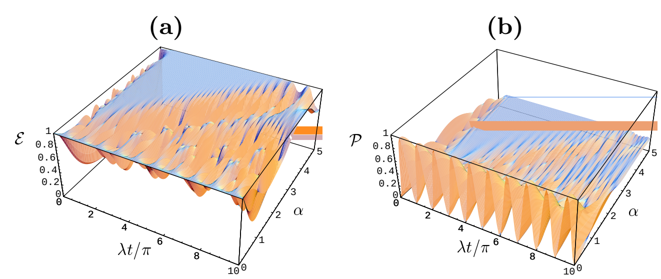

FIG. 1: **(a)**: Degree of entanglement for the cavity field depending on the interaction time λt (in unit of π) and the amplitude α of the initial coherent state; **(b)**: Probability of finding the atoms leaving the cavities in their ground states.

discrete nature to explain the revival of the oscillatory behavior in entanglement as shown in Fig. 1(a). The revival occurs when the sinusoidal functions in Eq. (7) are in phase. The significant contributions of the sinusoidal functions come from around the peak of the Poissonian distribution. At the peak of the revival time t_r: $2\lambda t_r\sqrt{\alpha^2} - 2\lambda t_r\sqrt{\alpha^2-1} = 2\pi$. Taking only the first two terms of the binomial expansion of the square root, we find the revival time $t_r = 2\alpha\pi/\lambda$. In fact, the dynamics of entanglement follows the well-known argument for the collapse and revivals of Jaynes-Cummings (JC) model [9].

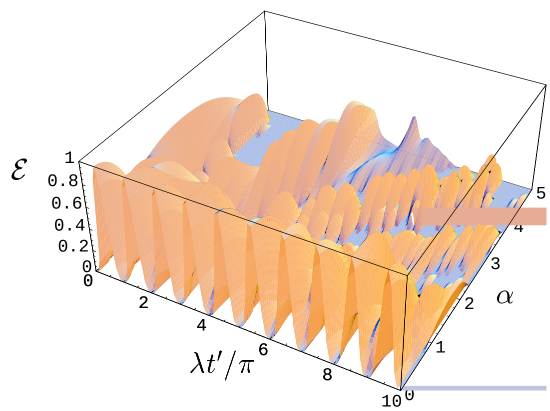

FIG. 2: Degree of entanglement for the second pair of atoms depending on the interaction time $\lambda t' = \lambda t$ and α.

Entanglement retrievals.- We have seen that the qubits can transfer a complete ebit to a CV system conditionally. The next question is 'Will it be possible for the qubits to retrieve the ebit from the CV system?' In order to solve this problem, we take the second set of atoms initially prepared in their ground states to send through the respective cavities which are in $|\psi(1)\rangle_f$. According to the earlier discussions on the propagation of the atom-field state, after the interaction time of t', the atom-field state becomes

$$|\Psi(2)\rangle_{a-f} = \begin{pmatrix} \hat{U}^{(1)}_{12}\hat{U}^{(2)}_{12} \\ \hat{U}^{(1)}_{12}\hat{U}^{(2)}_{22} \\ \hat{U}^{(1)}_{22}\hat{U}^{(2)}_{12} \\ \hat{U}^{(1)}_{22}\hat{U}^{(2)}_{22} \end{pmatrix} |\psi(1)\rangle_f = \sum_{n,m=0}^{\infty} \mathbf{V}_a |n\rangle_1 |m\rangle_2 \tag{10}$$

with the matrix

$$\mathbf{V}_a = \begin{pmatrix} -\sin(\lambda t'\sqrt{n+1})\sin(\lambda t'\sqrt{m+1})\mathcal{C}_{n+1,m+1} \\ -i\sin(\lambda t'\sqrt{n+1})\cos(\lambda t'\sqrt{m})\mathcal{C}_{n+1,m} \\ -i\cos(\lambda t'\sqrt{n})\sin(\lambda t'\sqrt{m+1})\mathcal{C}_{n,m+1} \\ \cos(\lambda t'\sqrt{n})\cos(\lambda t'\sqrt{m})\mathcal{C}_{n,m} \end{pmatrix}. \tag{11}$$

In order to investigate how much of the field entanglement deposited by the first set of atoms, would be transferred to the second set, we trace $|\Psi(2)\rangle_{a-f}$ over the field variables and find the state of the atoms: $\hat{\rho}_a = \sum_{n,m} \mathbf{V}_a \mathbf{V}_a^\dagger$. The degree of entanglement for the two atoms is found using the log negativity[8] of the partial transposition of the density operator $\hat{\rho}_a$ and plotted in Fig. 2 as a function of the interaction times $\lambda t'$. It is seen that, for non-vanishing α, the CV fields are not able to transfer the complete ebit to the atoms. However we cannot simply say that it is possible to transfer an ebit from a qubit system to a CV system while the converse is not true. The reason is that the qubit→CV transfer of an ebit was conditioned on the qubits having lost their entanglement completely. It is not straightforward to find such the condition on the CV state for the CV→qubit transfer.

In order to improve the degree of entanglement transferred to the atoms, we consider an orthogonal measurement of $\{\hat{P}^{(i)}_\alpha, \hat{Q}^{(i)}_\alpha = \mathbb{1} - \hat{P}^{(i)}_\alpha\}$, where $\hat{P}^{(i)}_\alpha$ is the projection onto the coherent state of its amplitude α. In Fig. 3, the degree of entanglement for the atoms is plotted, conditioned on the fields in $\hat{P}^{(1)}_\alpha \hat{P}^{(2)}_\alpha$ for interaction times $t' = t$. We can see the complete entanglement transfer for a CV system to a qubit system. This is analyzed as follows. By postselecting the event of (α, α) after the interaction time $t' = t$, the atomic state becomes

$$|\psi(2)\rangle_a = \frac{\mathcal{N}'}{\sqrt{2}} \begin{pmatrix} 2v_1v_2 \\ v_1v_4 + v_2v_3 \\ v_3v_2 + v_4v_1 \\ 2v_3v_4 \end{pmatrix}, \tag{12}$$

where $v_1 = \langle\alpha|\hat{U}_{12}\hat{U}_{21}|\alpha\rangle$, $v_2 = \langle\alpha|\hat{U}_{12}\hat{U}_{22}|\alpha\rangle$, $v_3 = \langle\alpha|\hat{U}_{22}\hat{U}_{21}|\alpha\rangle$, $v_4 = \langle\alpha|\hat{U}_{22}\hat{U}_{22}|\alpha\rangle$. $\mathcal{N}'$ is the new normalization factor. Using the same approximation leading to Eq. (9), we find that $v_{1,4} \approx \frac{1}{2}[\pm 1 + \cos(2\alpha\lambda t)\, e^{-\frac{(\lambda t)^2}{2}}]$, where $-$ sign is for v_1 and $+$ for v_4, and $v_2 = v_3 \approx \mathcal{O}(t)e^{-\frac{(\lambda t)^2}{2}}$ where $\mathcal{O}(t)$ is a linear sum of sinusoidal functions. In the long time limit $\lambda t \gg 1$, $v_2 = v_3 \to 0$ while $v_1 = -v_4 = -\frac{1}{2}$. State (12) is now a maximally entangled triplet state (1), which has been perfectly retrieved after the interactions with the cavity fields. This can be inferred from the analysis of the entanglement between the second pair of atoms shown in Fig. (3). It is surprising to note that the probability of getting the coherent state is as high as 50%, in this limit.

It is worth stressing that the results presented in this work are a feature of the coherent state used as a memory for entanglement. Indeed, we have checked that, by considering initial thermal states for the fields and applying the protocol in the previous paragraphs, the initial ebit in the qubit-state can never be completely deposited in the CV state and, consequently, retrieved from it. Obviously, in order to investigate the entanglement deposit, one faces the hard problem of quantifying the entanglement in a two-mode non-Gaussian state. This difficulty has been bypassed adopting the technique described in ref. [10] based on the projection onto a subspace spanned by the bidimensional bases $\{|n\rangle, |n+1\rangle\}_j$ with $j = 1, 2$.

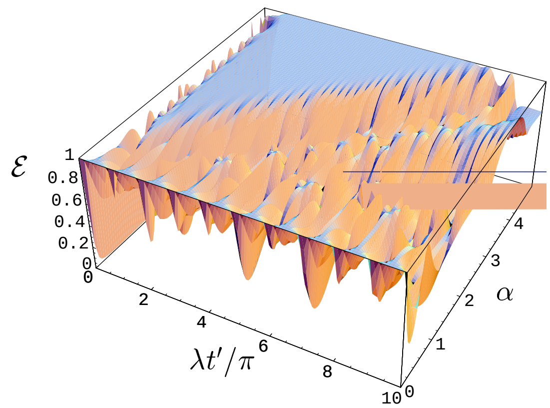

FIG. 3: Degree of entanglement for the second pair of atoms at the postselected event of (α,α), depending on the interaction time $\lambda t' = \lambda t$ and α. Note that the shape resembles very much Fig.1(a), the degree of entanglement for the cavity field.

The entanglement within the resulting projected state is then averaged over thermal weighting functions characterized by their mean photon number $\bar{n}$. This provides us with a lower bound to the entanglement in the two-mode non-Gaussian state. It is thus straightforward to see that the perfect deposit-retrieval process is possible just for the trivial case of $\bar{n} = 0$ for the initial fields.

An interesting question to ask now is what happens when a series of atom pairs, each in the state $\psi(0)\rangle_a$, are allowed to interact with the cavity fields (in the usual setting of one atom with each cavity field). We found that, for example, for $\alpha = 4.5$ and if the first pair of atoms had interacted for a time $t_1 = 6.47/\lambda$ (which deposits an ebit of entanglement), then a second pair of atoms interacting for a time $t_2 = 11.04/\lambda$ deposits another ebit, and a third pair interacting for a time $t_3 = 3.24/\lambda$ deposits yet another ebit. Each of these depositions have a success probability of ~ 0.25 and are robust to small variations in t_i, as before. This contrasts the case of the cavities starting in vacuum states where incommensurate Rabi frequencies prevent the deposition of more than one ebit through the resonant interaction. The cavities in our case can thus serve as "stationary" reservoirs for multiple ebits supplied by atom pairs in the form of "flying" qubits, which may be difficult to hold in other situations. In addition, these multiple ebit entangled cavity states may be directly used for teleportation of higher dimensional states. Application of $\hat{P}_\alpha$, $\hat{Q}_\alpha$ also allows the retrieval of 1.82 ebits and 1.91 ebits at optimized times from the 2- and 3-ebit entangled cavity states respectively through a pair of atoms.

As a final remark, we would like to shortly point out that our approach is quite setup-independent. Obviously, an implementation based on quantum electrodynamics in cavity would be the most natural choice [11]. However, the interaction model we have assumed, the resonant JC one, turns out to be naturally valid in many physical situations in which coherent exchange of excitations between spin-like particles and bosons are involved [12]. The very recent progress in micro and nano-fabrication of integrated cavity-qubit systems in the semiconductor and superconducting domain [13] and the readily available sources of coherent states in many ranges of frequency makes our proposal adaptable to different physical situations. The language we have adopted in this paper, thus, has to be seen as a pure matter of convenience.

Remarks.- In this paper, we have considered interface between two hetero-dimensional systems. An ebit can be transferred to a CV system from a qubit system and back to the qubit system conditionally. One extremely nice thing is that the transfer happens in the quasi-steady state, which means that one does not have to be careful in picking the time for entanglement transfer. We also found an interesting analogy between the entanglement reciprocation and the collapse and revival of Rabi oscillations in the JC model considered for the proof of discreteness in the photon number. The perfect reciprocation of entanglement is a particular feature of a coherent state. Postselecting the fields in (α,α) is not trivial. Even though a heterodyne detection or a beam splitter detection may approximate it, this deserves a further investigation.

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